Mechanics is the branch of applied science dealing with forces and motions. Fundamental to the field is the notion of equilibrium, the condition existing when a system of forces acting on a body is in a state of balance. The term statics is used to describe the part of mechanics specifically concerned with relations between forces acting on rigid bodies that are in equilibrium and at rest. The term dynamics refers to the part of mechanics dealing with rigid bodies in motion. If correctly placed inertial forces are taken into account, bodies in motion can also be considered to be in equilibrium. The field of study usually referred to as strength of materials is an extension of mechanics that addresses the relationship between applied or external forces acting on a body and the internal effects produced by those forces in the body. The study of the deformations produced in a body by a set of external forces is an integral part of the field of strength of materials.

The distinctions made here reflect the way the study of the general subject area has evolved in the engineering disciplines. In most engineering curricula, statics, dynamics, and strength of materials are normally treated as separate topics presented sequentially under the general umbrella of mechanics. During the process of analyzing and designing structures in buildings, however, professionals freely use ideas and elements from each of these basic fields (as well as others) as tools in a nonsequential manner. The chapters in Part II adopt this more integrative approach. At this point, however, it is pedagogically useful to retain the traditional engineering distinctions.

In the first part of this chapter, we introduce certain fundamental ideas in statics. In the second part, we focus on basic elements of strength of materials. The field of dynamics is outside the scope of the book and is not addressed. The following presentation is intended to provide only an overview of the basic issues involved in statics and the strength of materials.
Topics are therefore presented succinctly. The reader is referred to any of a number of basic texts that treat the subject matter more elaborately.¹

**Note to readers and instructors:** Although presented in a logical continuum of topics, the reader may not wish to not cover all of the topics in the order in which they are presented, since they are often fairly abstract and may be better understood when covered in parallel with material in subsequent chapters. The material on statics, for example, forms a needed prelude to subsequent chapters on load modeling, trusses, cables, and arches. It is not absolutely necessary to know everything about shear and moment diagrams or material properties that are presented later in the chapter in order to understand the essence of these topics, but some understanding of them is essential for dealing with beams and columns. Depending on the curricula, for example, topics such as shear moment diagrams are often best covered after truss analysis and before beam analysis. The topics in this chapter are thus presented to be drawn from as needed. A reasonable alternative sequencing of topics is noted in the preface to the book.

2.2 Forces And Moments

2.2.1 Analysis Objectives and Processes

Section 2.2 explores forces, moments, and equilibrium concepts. Before developing these concepts in a formal way, however, it is useful to look more directly at a simple example to show where the discussion will lead.

Concepts of force, moment, and equilibrium are given visual expression in the mobile by Alexander Calder illustrated in Figure 2.1.² Clearly, the mobile as a whole is in careful balance, as is each of its parts. Any typical arm of the mobile experiences a set of forces acting upon it. These forces consist of both external, or applied, forces—the weights—and internal forces, or reactions, that are developed within the structure at connection points. The arm itself must be in equilibrium with respect to the various forces (assuming that it is not sliding in space or spinning madly about). The net translational effect of all forces or their components acting on an object must have a sum of zero along any axis or in any direction. As developed in Section 2.3, this fundamental requirement will be formally expressed in the following way: \( \Sigma F = 0 \) (a symbolic notation read as “the sum of forces equals zero”). Likewise, there must be no net rotational effects of the forces about the point of suspension. Note that these rotational effects can be quantified as a product of the magnitude of the force times its distance from the point of suspension \( F \times d \). Such a rotational effect is called a moment \( M = F \times d \). Moments can act in clockwise or counterclockwise directions. For the arm to be in rotational equilibrium, the net total of the rotational moments acting on it must be zero, or \( \Sigma M = 0 \). In the mobile, a clockwise moment acting on an arm is invariably exactly balanced by a moment that acts in a counterclockwise direction. Note that a small force acting a long distance away from the suspension point can have the same rotational effect as a large force acting over a short distance.

¹Students wishing a more extensive coverage of the application of structural principles to sculpture should see Schodé, *Structure in Sculpture*, M.I.T. Press, Cambridge, Massachusetts, 1993.
Figure 2.1 Basic equilibrium diagrams for each arm of the mobile are shown to the right. The diagram for arm D is shown in more detail.

The analytical drawings which illustrate the force systems that act on an object are called equilibrium diagrams (sometimes also called free-body diagrams; see Section 2.3.3). Constructing these kinds of diagrams is a first step in making a static analysis of a structure. Using equilibrium concepts, it is then possible to determine numerical values for the reactions (force interactions generated by the action of one object on another; see Section 2.3.3) that occur at supports or connections. Subsequently, internal forces (shears and bending moments; see Section 2.4.3), stresses (internal forces per unit area), and strains (deformations caused by stresses) can be determined (see Section 2.6).

The following sections explore these concepts in greater detail. What happens, for example, if the forces applied to an object are inclined? What if the forces are distributed over a surface rather than acting at a point?

2.2.2 Forces

Fundamental to the field of mechanics is the concept of force and the composition and resolution of forces. A force is a directed interaction between bodies. Force interactions have the effect of causing changes in the shape or motion (or both) of the bodies involved. The basic concept of force is undoubtedly familiar to the reader and is probably felt to be intuitively
obvious. Viewed from a historical perspective, however, the idea of force and the characterization of a force in terms of magnitude, sense, and direction was hardly obvious. The precise formulation of these concepts is really quite a remarkable accomplishment in view of the degree of abstraction involved. Indeed, the distinction between force and weight, as well as the notion of a nonvertical force, was only just beginning to be appreciated by scholars in the Middle Ages. The name Jordanus de Nemore is repeatedly connected to the emergence of these concepts. Once force was conceived in vectorial (directional) terms, the problem of the components of a force and the general composition and resolution of forces was addressed by a variety of individuals, including Leonardo da Vinci, Stevin, Roberval, and Galileo. This problem, often termed the basic problem in statics, was finally solved by Varginon and Newton.

2.2.3 Scalar and Vector Quantities

A distinction is invariably made in the study of mechanics between scalar and vector quantities. Scalar quantities can be adequately characterized by magnitude alone. Vector quantities must be characterized in terms of both magnitude and direction. Forces are typically vector quantities. Any vector quantity can be represented by a line. The direction of the line with respect to a fixed axis denotes the direction of the quantity. The length of the line, if drawn to scale, represents the magnitude of the quantity. [See Figure 2.2(a).]

The line of action of a force is a line of indefinite length, of which the force vector is a segment. Since most structures that will be dealt with in this text are essentially rigid bodies that deform only slightly under the application of a force, it can be assumed that the point of application of a given force may be transferred to any other point on the line of action without altering the translatory or rotational effects of the force on the body. Thus, a force applied to a rigid body may be regarded as acting anywhere along the line of action of the force.

2.2.4 Parallelogram of Forces

Essential to a study of structural behavior is knowing the net result of the interaction of several vector forces acting on a body. This interaction can be studied in terms of the laws of vector addition. These laws and fundamental postulates are based on experimental observation. Historically, the first method of adding vector quantities was based on the parallelogram law. In terms of force vectors, the law states that when the lines of action of two forces intersect, there is a single force, or resultant, that is exactly equivalent to the two forces and that can be represented

![Figure 2.2](image-url)  
Figure 2.2 Free vectors, force interactions, resultant forces, and the parallelogram of forces.
by the diagonal of the parallelogram formed by using the force vectors as its sides. [See Figure 2.2(b)-(d).] In general, a resultant force is the simplest force system to which a more complex set of forces may be reduced and still produce the same effect on the body acted upon. Figure 2.3 illustrates a numerical example.

A graphic technique for finding the resultant force of several force vectors whose lines of action intersect is illustrated in Figure 2.4. The individual vectors, drawn to scale, are joined in tip-to-tail fashion. The order of combination is not important. Unless the resultant force is zero, the force polygon thus formed does not form a closed figure. The closure line is identical to the resultant force of the several individual vectors (i.e., the resultant is that vector which would extend from the tail of the first vector to the tip of the last vector in the group). The resultant closes the force polygon. This general technique follows from the parallelogram law. An algebraic method for finding the resultant of several forces acting through a point is discussed in Section 2.2.6.

Although conceptually very simple, graphical approaches to finding the resultants of force systems are extremely powerful as structural analysis aids. Historically, they found wide usage because of the ease of their application. Graphical techniques were used extensively by early investigators in their attempts to understand the behavior of complex structures. Figure 2.5 illustrates a latter-day analysis of a gothic structure by graphic techniques. Figure 2.6 illustrates the sphere model of an arch that utilizes graphic techniques. The model, based simply on the parallelogram law, is still a very elegant way of looking at arches. Although going into

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**Figure 2.3** Parallelogram methods of finding the resultant force $R$ of two concurrent forces. $R$ can be found algebraically or graphically.

**Figure 2.4** Graphical methods of finding the final resultant force $R_B$ of a concurrent force system.
Figure 2.5 Application of graphic methods to the analysis of a gothic structure (Amiens). The dead weights of the vertical buttresses and pinnacle help turn the thrusts of the flying buttresses downward through the middle portion of the buttresses themselves. The vertical buttresses are stable and not prone to overturning or cracking when the force resultants pass through the middle portions of these buttresses.

either of these analyses in detail is beyond the scope of the book, it can be noted that the techniques used are extremely simple and based on the concepts discussed in this chapter. Graphical techniques are no longer used extensively, but remain an elegant way of looking at structures and are useful in developing an intuitive feeling for the flow of forces in a structure.

2.2.5 Resolution and Composition of Forces

A process that follows directly from the fundamental law concerning the parallelogram of forces is that of breaking up a single force into two or more separate forces that form a system of forces that is equivalent to the initial force. This process is usually referred to as resolving a force into its components. The number of components that a single force can be resolved into is limitless. (See Figure 2.7.) In structural analysis, it is often most convenient to resolve a force into rectangular, or Cartesian, components. By utilizing right angles, components can be found by means of simple trigonometric functions. When a force $F$ is resolved into components on the $x$- and $y$-axes, the components become $F_x = F \cos \theta$ and $F_y = F \sin \theta$. The process is reversed if $F_x$ and $F_y$ are given and it is desired to know the resultant force $F$: $F = \sqrt{F_x^2 + F_y^2}$ and $\theta = \tan^{-1}(F_y/F_x)$.

Note that using Cartesian components is a matter of convenience only. A right angle is nothing more than a special form of a parallelogram. Figure 2.7 illustrates several manipulations with components. For example, a force of $F = 500$ lb that acts $30^\circ$ to the horizontal has components $F_x = 500 \cos 30^\circ = 433$ lb and $F_y = 500 \sin 30^\circ = 250$ lb. Similarly, the resultant of two forces of 100 and 200 lb acting orthogonally to one another is given by
(a) From Poleni, *Memorie istoriche della Gran Copola del Tempio Valicano*, 1748: The spheres are arranged in accordance with the line of thrust.

(b) The line of thrust is the locus of the internal force resultant developed in the structure.

*Figure 2.6* Early "sphere model" of an arch: If a series of spheres is stacked as illustrated the assembly is stable. Note that the shape is the inversion of a freely hanging chain made of similar spheres. The assembly will collapse if either the loading or the positioning of the spheres is changed.

(a) Force vector.

(b) Components of force on m-n axes.

(c) Components of force on x-y axes.

*Figure 2.7* Resolution of a force into components.
\[ R = \sqrt{F_x^2 + F_y^2} = \sqrt{(100)^2 + (200)^2} = 223.6 \text{ lb}, \quad \text{with} \quad \theta = \tan^{-1}(100/200) = 26.6^\circ. \]

Components obviously could be found in three dimensions as well as two.

**EXAMPLE**

Determine the components on the \( x \)- and \( y \)-axes of a force \( F \) of 1000 lb that acts at an angle of \( \phi = 60^\circ \) to the \( x \)-axis.

**Solution:**

\[ F_x = F \cos \phi = 1000 \cos 60^\circ = 500.0 \text{ lb} \]
\[ F_y = F \sin \phi = 1000 \sin 60^\circ = 866.6 \text{ lb} \]

Alternatively, if the components of a force \( F \) were known to be orthogonal and have values of 866.6 lb and 500 lb on the \( x \)- and \( y \)-axes, respectively, you would find the magnitude and direction of \( F \) with the formulas

\[ F = \sqrt{F_x^2 + F_y^2} = \sqrt{(500.0)^2 + (866.6)^2} = 1000 \text{ lb} \]

and

\[ \tan \phi_x = \frac{500}{866.6} = 0.5773 \quad \therefore \quad \phi_x = 60^\circ \]

or

\[ \phi_x = \tan^{-1} \frac{866.6}{500} = 60^\circ \]

and

\[ F = F_x / \cos \phi = 500 / \cos 60^\circ = 1000 \text{ lb} \]

### 2.2.6 Statically Equivalent Systems

Implicit in the discussion in the previous section is the notion of static equivalency. When a system of forces applied to a body can be replaced by another system of forces applied to the same body without causing any net change in translational or rotational effects on the body, the two force systems are said to be **statically equivalent**. A resultant force, for example, is statically equivalent to the force system from which it was derived. The series of diagrams in Figure 2.8 illustrate a graphical process for determining the statically equivalent single resultant force of a series of coplanar concurrent forces. If an equilibrating force \( F \) that is exactly equal and opposite to the resultant force \( R \) is applied to the same point, then the point is not subjected to any net force. (The point is said to be in equilibrium; see next section.) The algebraic process illustrated depends on resolving each force into components (\( F_x \) and \( F_y \) forces) and summing components acting in the same direction. The resultant force is then given by \( R = \sqrt{\left( \Sigma F_x \right)^2 + \left( \Sigma F_y \right)^2} \) and its orientation by \( \theta_x = \tan^{-1} \left( \Sigma F_y / \Sigma F_x \right) \).

**Concurrent forces** act through the same point and thus do not tend to produce rotational effects about that point. (The moment arms are zero.) A single statically equivalent force for a nonconcurrent force system (in which forces do not intersect at a common point) can also be found. This process is illustrated in Appendix 2.

### 2.2.7 Moments

**Moment of a Force.** A force applied to a body tends to cause the body to translate in the direction of the force. Depending on the point of application of the force on the body, however, the force may also tend to cause the body to rotate. This tendency to produce rotation is
called the moment of the force. (See Figure 2.9.) With respect to a point or line, the magnitude of this turning or rotational tendency is equal to the product of the magnitude of the force and the perpendicular distance from the line of action of the force to the point or line under consideration. The moment $M$ of a force $F$ about a point $O$ is thus simply $M_O = F \times r$, where $r$ is the perpendicular distance from the line of action of $F$ to $O$. The distance $r$ is often called the moment arm of the force. A moment has the units of force times distance (e.g., ft-lb or N·m). A force of 1000 lb acting 5 ft away from a point produces a moment of $M = F \times r = 1000 \text{ lb} \times 5 \text{ ft} = 5000 \text{ ft-lb}$. 

**Figure 2.8** Resultants and equilibrating forces.
EXAMPLE

The concept of a moment can be used to understand the behavior of many familiar objects. In the Calder mobile (Figure 2.1), the moments on each side of the point of suspension must be identical in order for balance to occur. Assume that \( W = 100 \text{ lb} \), \( F_E \) (the weight of the lower arm) = 150 lb, \( a_1 = 4.5 \text{ ft} \), and \( a_2 = 3 \text{ ft} \). Demonstrate that the arm is in balance about the point of suspension. (See Figure 2.10.)
Solution: The rotational moment of \( W \) about the point of suspension is given by

\[
M_1 = W \times a_1 = 100 \text{ lb} \times 4.5 \text{ ft} = 450 \text{ ft-lb}
\]

This rotational effect must be balanced by the moment produced by the other force:

\[
M_2 = 150 \text{ lb} \times 3 \text{ ft} = 450 \text{ ft-lb}
\]

Since \( W \times a_1 = F_E \times a_2 \), the arm is in rotational balance (\( \Sigma M = 0 \)).

Moment of a Distributed Load. Loads are frequently uniformly distributed along the length of a member (e.g., \( w \) lb/ft). The total force on the member equivalent to this uniformly distributed load is simply \( wL \). The moment of the distributed force can then be found by imagining the uniformly distributed load concentrated into an equivalent point load of magnitude \( wL \) located at the point of symmetry of the loading and multiplying it by the distance to this point \( (L/2) \) for a uniform load). The process is illustrated in Figure 2.9. If a triangular load distribution (which varies from zero to a maximum value of \( w \)) is present, the equivalent concentrated load is \( wL/2 \), and it acts at the two-thirds point of the loading. Modeling loads in this way is valid only for purposes of calculating reactions, not for constructing shear and moment diagrams or for other purposes. For example, for the member shown in Figure 2.11, a uniform loading of 100 lb/ft on a 20-ft-long member produces a moment about point \( O \) of \( M_O = (wL)(L/2) = (100 \text{ lb/ft})(20 \text{ ft})(10 \text{ ft}) = 20,000 \text{ ft-lb} \).

The foregoing method is too simple and is difficult to apply when loads are more complex and when they vary along the length of a member. Appendix 3 describes a more general and more powerful technique for determining moments of varying loads of any kind.

Moments Due to Multiple Forces. The total rotational effect produced by several forces about the same point or line is merely the algebraic sum of their individual moments about that point or line. Thus, \( M_O = (F_1 \times r_1) + (F_2 \times r_2) + \cdots + (F_n \times r_n) \).

Moments about a Line. The rotational effect on a rigid body caused by multiple forces acting about a line, but not in the same plane, is the same as that which would result if all of the forces were acting in the same plane.

Moment of a Couple. A couple is a force system made up of two forces equal in magnitude, but opposite in sense, and with parallel lines of action that are not on the same straight line (\( \perp \)). A couple tends to cause only rotational effects on bodies and does not cause translation. The moment of a couple is simply the product of one of the forces and the perpendicular distance between the two forces. It can be shown, however, that the moment of a couple is actually independent of the reference point selected as a moment center. The magnitude of the rotational effect produced by a couple on a body is also independent of the point of application of the couple on the body.
2.3 Equilibrium

2.3.1 Equilibrium of a Particle

A body is in equilibrium when the force system acting on it tends to produce no net translation or rotation of the body. In effect, the body is in a state of balance. Equilibrium exists in concurrent force systems (systems in which all forces act through a single point) when the resultant of the force system equals zero. A concurrent force system having a nonzero resultant force may be put in equilibrium by applying another force (typically called an equilibrant) that is equal in magnitude and on the same line of action, but of opposite sense.

If a force system is in equilibrium, its resultant must be zero, and it follows that $\Sigma F_x = 0$ and $\Sigma F_y = 0$. Thus, the algebraic sum of all the components of the forces applied to a particle in the $x$ direction must be zero, and likewise for the $y$ direction. Note that $x$ and $y$ need not be horizontal and vertical, respectively. The previous statement is true for any orthogonal set of axes, no matter what their orientation.

More generally, the conditions $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma F_z = 0$ are necessary and sufficient to ensure equilibrium in a concurrent force system. A force system satisfying these conditions will not cause the particle to translate. (Rotation is not a problem, since all forces act through the same point in concurrent force systems).

2.3.2 Equilibrium of a Rigid Member

General Equilibrium Conditions. When a nonconcurrent force system acts on a rigid body, the potential for both translation and rotation is present. For the rigid body to be in equilibrium, neither must occur. With respect to translation, this implies that, as in the case of concurrent force systems, the resultant of the force system must be zero. With respect to rotation, the net rotational moment of all forces must be zero. The conditions for equilibrium of a rigid body are, therefore, $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma F_z = 0$ and $\Sigma M_x = 0$, $\Sigma M_y = 0$, $\Sigma M_z = 0$.

Sign Conventions. In working with general force systems, sign conventions are always problematic. One classic approach to length measurements is to use $x$, $y$, and $z$ values measured from a reference point on the extreme left of the structure. Thus, distances measured from this point to the right are always positive (+$x$). For forces, a similar convention can be used. A force acting vertically upward would be positive (+$F$) and one acting downward would be negative (−$F$). Components of forces would be similarly designated. For rotational moments, note that with respect to a reference point on the extreme left of the structure, an upward force (+$F$) located to the right a positive distance (+$d$) would cause a positive moment about the point that acts in a counterclockwise direction, or $M = (+F)(+d) = +Fd$. A downward force would be negative (−$F$) and cause a negative moment that acts in the clockwise direction, or $M = (−F)(+d) = −Fd$. While useful, this convention can get cumbersome and confusing (particularly as regards rotational moments) when nonvertical forces are present. Most individuals simply use a convention directly relating to the rotational sense of the moment effect. Thus, moments that tend to produce counterclockwise rotations are considered positive ($\theta M$), and those that tend to produce clockwise rotations are considered negative ($\Theta M$). These conventions are for equilibrium calculations involving external forces only. Other conventions will be developed later for describing internal forces and moments that act within the structure.
EXAMPLE

Determine the unknown forces $F_A$ and $F_B$ in the structure shown in Figure 2.11 so that it is in a state of static equilibrium.

**Figure 2.11**

```
\[ \begin{array}{c}
4P \\
\downarrow \\
15 \\
\downarrow \\
F_A \\
\downarrow \\
5 \\
\downarrow \\
F_B \\
\downarrow \\
4P \\
\downarrow \\
1P \\
\downarrow \\
3P \\
\end{array} \]
```

**Solution:**

*Force equilibrium in the vertical direction, $\Sigma F_y = 0$: $F_A + F_B - 4P = 0$.*

This expression cannot be solved yet, since we have only one equation and two unknown force values.

*Force equilibrium in the horizontal direction, $\Sigma F_x = 0$:*

No forces act in the horizontal (or x) direction.

*Moment equilibrium about point A using force and distance algebraic conventions*

\[
\Sigma M_A = 0 \\
( + F_A)(0) + (-4P)(+15) + (+ F_B)(+20) = 0 \\
F_A(0) - (4P)(15) + F_B(20) = 0 \\
\therefore F_B = 3P \uparrow
\]

*Moment equilibrium about point A using counter-clockwise rotations as positive*

\[
\Sigma M_A = 0 \\
-4P(15) + F_B(20) = 0 \\
\therefore F_B = 3P \uparrow
\]

From $\Sigma F_y = 0$: $F_A + F_B - 4P = 0$:

\[
F_A + 3P = 0 \\
\therefore F_A = 1P \uparrow
\]

In the next section, we will see that the forces $F_A$ and $F_B$ in situations similar to that shown typically occur at structural supports and are reactive in nature. By contrast, there might be situations where the forces are applied. Here, the point is that no matter what their origin, they must have the numerical values noted in order for the structure to be in equilibrium.

**Two-Force Members.** Several special equilibrium cases exist. When a rigid member is subjected to only two forces, the forces cannot have arbitrary magnitudes and lines of action if the member is to be in equilibrium. Consider the member shown in Figure 2.12. By summing moments about point A, it can be seen that there is no way that the structure can be in rotational
equilibrium unless the line of action of the force at B passes through point A. In a similar way, it is necessary that the line of action of the force at A pass through point B if the structure is to be in rotational equilibrium about point B. Thus, the two forces must be collinear. They must also be equal in magnitude, but opposite in sense. This result is particularly important in the analysis of trusses.

Three-Force Members. As with two-force members, three forces acting on a member cannot have random orientations and magnitudes if the member is to be in equilibrium. This can be seen with reference to Figure 2.12. For the member to be in rotational equilibrium, the lines of action of all three forces must pass through a common point.

Figure 2.13 shows a latter-day analysis of a gothic structure which illustrates the principle that the lines of action of three-force members must pass through the same point.

2.3.3 Applied and Reactive Forces

Forces and moments that act on a rigid body can be divided into two primary types: applied and reactive. In common engineering usage, applied forces are forces that act directly on a structure (e.g., the force produced by snow). Reactive forces are forces generated by the action of one body on another and hence typically occur at connections or supports. The existence of reactive forces follows from Newton’s third law, which, broadly speaking, states that, to every action, there is an equal and opposite reaction. More precisely, the law states that whenever one

Figure 2.13 Application of graphical methods to the analysis of a gothic structure. The weight of the buttress and its reactions form a three-force system. Consequently, all forces meet at a single point.
body exerts a force on another, the second always exerts a force on the first that is equal in magnitude and opposite in direction and that has the same line of action. In Figure 2.14(b), the force on the beam causes downward forces on the foundation, and upward reactive forces are consequently developed. A pair of action and reaction forces thus exists at each interface between the beam and its foundations. In some cases, moments form part of the reaction system as well. [See Figure 2.14(c).]

The diagrams in Figure 2.14, which show the complete system of applied and reactive forces acting on a body, are called free-body diagrams.

If a body (such as any of those illustrated) is indeed in a state of equilibrium, the general conditions of equilibrium for a rigid body that were stated in the previous section must be satisfied. The magnitude and direction of any reactive forces that are developed must be such that equilibrium is maintained and are thus necessarily dependent on the characteristics of the applied force system. The whole system of applied and reactive forces acting on a body (as represented by the free-body diagram) must be in a state of equilibrium. Free-body diagrams are, consequently, often called equilibrium diagrams. Drawing equilibrium diagrams and finding reactions for loaded structural members is a common first step in a complete structural analysis.
<table>
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<th>Type of connection</th>
<th>Typical symbols</th>
<th>Types of translations and rotations that the connection allows</th>
<th>Type of forces that can be developed at the connection</th>
<th>Types of forces that can be developed when the support is inclined</th>
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</thead>
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<tr>
<td>Pinned support</td>
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<tr>
<td>Roller support</td>
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<tr>
<td>Cable support</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
</tr>
</tbody>
</table>

Figure 2.15 Types of support conditions: idealized models.

**Support Conditions.** The nature of the reactive forces developed on a loaded body depends on the exact way in which the body is either supported by or connected to other bodies. Figure 2.15 illustrates relations between the type of support condition present and the type of reactive forces developed. Several basic types of support conditions are indicated; others are possible. Of primary importance are pinned connections, roller connections, and fixed connections. In *pinned connections*, the joint allows attached members to rotate freely, but does not allow translations to occur in any direction. Consequently, the joint cannot provide moment resistance, but can provide force resistance in any direction. A *roller connection* also allows rotations to occur freely. It resists translations, however, only in the direction perpendicular to the face of the support (either into or away from the surface). It does not provide any force resistance parallel to the surface of the support. A *fixed joint* completely restrains rotations and translations in any direction. Consequently, it can provide moment resistance and force resistance in any direction. Other types of supports include a *cable support* and a *simple support*. These are similar to the roller connection, except that they can provide force resistance in one direction only. The connections shown are, of course, idealized. *The relation between these idealized connections and those actually present in building structures is discussed in Section 3.3.2. The reader should study Section 3.3.2 in parallel with the calculation principles to be presented.*
For a structure to be stable, the supports must provide a specific minimum number of force restraints. For a simple beam loaded with both downward and horizontal forces, there must be three such restraints, corresponding to the fact that there are three conditions of equilibrium for that type of structure: $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M = 0$. One way of meeting this requirement is to use a fixed connection. Another is to use a pinned connection on one end and a roller connection on the other. It is, of course, possible to use connections that offer more degrees of restraint than the minimum required.

Structures having connections or supports that provide more than the minimum needed for stability are referred to as *statically indeterminate externally*. Since there are more unknown constraining forces than there are equations of equilibrium, it is not possible to solve for the magnitudes of these constraining forces by statics alone. Other techniques, discussed in succeeding chapters, must be used instead.

**Reactions for Typical Load and Support Conditions.** The examples that follow illustrate how reactions are calculated for common situations. In the first example, the reactions must act only in the vertical direction, since the beam is sitting on just two supports. (Note that the beam would not be stable and would tend to slide if a horizontal force were applied to it.) In the second example, the applied loads act vertically downward, and the structure has pin and roller support conditions. In the preceding discussion, it was argued that a reactive force can develop in any direction at a pinned connection, but only perpendicular to the plane of support in a roller joint. Common sense correctly suggests, however, that when all applied loads act vertically, all reactive forces, including the one at the pin, will also act vertically. From a calculational perspective, it is still necessary to assume initially that the reaction at the pin is inclined. The reaction analysis should then predict that the force is indeed vertical. (The horizontal component should turn out to be zero, meaning that the force actually acts vertically.) In the third example, the applied force is inclined. An inclined reactive force then develops at the pin. This is a common result because, when a structure carries any applied force with a horizontal component, the reactive force at the pin will also be inclined so that necessary equilibrium balances can be obtained. The examples include cantilever structures, which illustrate that roller condition can develop either compressive or tension (tie-down) reactions.

**EXAMPLE**

Determine the reactions $R_A$ and $R_B$ for the beam shown in Figure 2.16, which merely rests on two supports.

![Figure 2.16 Reactions of a beam resting on two supports.](image-url)
**Solution:** The reactions must act only in the vertical direction for these support conditions. Since the loading is not symmetrical, it is necessary to determine one of the reactions by moment equilibrium first. Point A is chosen for convenience. Counter-clockwise moments are assumed to be positive. The other reaction is found by vertical force equilibrium. A static check is shown to verify results.

**Moment equilibrium about Point A:**

\[ \Sigma M_A = 0: \quad -9 \text{ ft} \times 4000 \text{ lb} - 21 \text{ ft} \times 8000 \text{ lb} + (0)R_A - 30R_B = 0 \]

\[ \therefore \quad R_B = 6800 \text{ lb} \]

**Force equilibrium in the vertical direction:**

\[ \Sigma F_y = 0: \quad -4000 \text{ lb} - 8000 \text{ lb} + R_A + R_B = 0 \]

\[ -12000 + R_A - 6800 = 0 \]

\[ \therefore \quad R_A = 5200 \text{ lb} \]

**Check: Moment equilibrium about Point A:**

\[ \Sigma M_B = 0: \quad +21 \text{ ft} \times 4000 \text{ lb} + 9 \text{ ft} \times 8000 \text{ lb} + (0)R_B - 30R_A = 0 \]

\[ \therefore \quad R_B = 5200 \text{ lb} \]

---

**EXAMPLE**

Determine the unknown reaction forces \( R_A \) and \( R_B \) in the structure shown in Figure 2.17. Demonstrate that the horizontal component of \( R_B \) is zero for this loading. Use \( x \) and \( y \) components of \( R_B \) in the equilibrium analysis.

**Figure 2.17** Reactions for a simple beam with vertical loads.

**Solution:** The reactive force roller \( R_A \) at the left must, by definition, act perpendicularly to the support plane. The pinned condition on the right can provide a reactive force resistance \( R_B \) in any direction, so it is shown acting at an angle and broken into \( x \) and \( y \) components. By inspection, however, and noting that the external forces act only in the vertical direction, it can be simply surmised that the actual direction of the reactive force at the pin is in the vertical direction only. If there were a horizontal force component present, the structure could not be in equilibrium in the vertical direction. Use force components for \( R_B \) in the following analysis:
Force equilibrium of all the forces acting in the vertical direction, $\Sigma F_y = 0$ \[ R_A + R_{B_y} = 1P - 4P = 0 \] or \[ R_A + R_{B_y} = 5P \]

While valid, this equation cannot yet be solved, due to the presence of two unknowns.

**Force equilibrium in the horizontal direction, $\Sigma F_x = 0$**: \[ 0 + R_{B_x} = 0 \]

Since there are no forces acting in the horizontal direction, $R_{B_x}$ must obviously be zero, which in turn implies that the resultant $R_B$ acts vertically and is identically equal to $R_{B_y}$.

**Moment equilibrium about point A, $\Sigma M_A = 0$**: The common convention that moments acting in the counterclockwise direction are positive is used.

\[
R_A(0) - (1P)(5) - 4P(15) + 20R_{B_y} + R_{B_x}(0) = 0
\]

or \[-5P - 60P + 20R_{B_y} = 0\] \[R_{B_y} = 3.25P\]

Note that, since $R_{B_x} = 0$, $R_B = R_{B_y}$.

\[
R_A + 3.25P = 5P \
\therefore \quad R_A = 1.75P
\]

---

**EXAMPLE**

Determine the reactions to the structure in Figure 2.18. The structure has an inclined applied force.

**Figure 2.18**

Inclined load = $4P \oplus 60^\circ$

Horizontal component \[ = 4 \cos 60 = 2.0P \]

Vertical component \[ = 4 \sin 60 = 3.46P \]

A \[ \downarrow \]

5 ft \[ \rightarrow \]

B \[ \uparrow \]

$R_A$ at roller must act perpendicular to the support

$R_B = R_B \cos \theta_B$

$\theta_B$

$R_B$ at pin can act in any direction

Inclined load = $4P \oplus 60^\circ$

$R_A = 2.0P$

$R_B = 1.73P$

$R_B = 2.64P$

$\theta = 40.85^\circ$

$R_B$ is found from resolving $R_{B_x}$ and $R_{B_y}$ into a single resultant force

**Solution**: For convenience, the applied load is broken up into horizontal and vertical components. These components are then considered as applied loads. The vertical reactions are found first. Since $\Sigma F_y = 0$ can rarely be directly solved first, use $\Sigma M_A = 0$, which will immediately yield the vertical component of the reactive force at $B$. 

---
Moment equilibrium about point A, \( \sum M_A = 0 \) \( \Rightarrow \):

\[
R_A(0) - 4P \sin 60^\circ(5) + R_B_x(10) + R_B_y(0) = 0
\]

or \(-3.46P(5) + 10R_B_x = 0\) \( \Rightarrow \) \( R_B_x = 1.73P \)

\( R_{B_y} \) is the component of the reactive force at \( B \) in the vertical direction. To find the reactive force at \( A \) that acts in the vertical direction—which must act vertically because of the roller condition—sum forces in the vertical direction.

Force equilibrium of all the forces acting in the vertical direction, \( \Sigma F_y = 0 \) \( \Rightarrow \):

\[
-4P \sin 60^\circ + R_{A_y} + R_{B_y} = 0
\]

\[-3.46P + R_{A_y} + 1.73P = 0\] \( \Rightarrow \) \( R_{A_y} = 1.73P \)

Force equilibrium in the horizontal direction, \( \Sigma F_x = 0 \) \( \Rightarrow \):

\[
4P \cos 60^\circ - R_{B_x} = 0
\]

\[2.0P - R_{B_x} = 0\] \( \Rightarrow \) \( R_{B_x} = 2.0P \)

Resultant reactive force \( R_B \) at \( B \):
To find the resultant force at \( B \), use the two known components, \( R_{B_y} \) and \( R_{B_x} \):

\[
\tan^{-1} \theta_B = 1.73/2.0 \quad \Rightarrow \quad \theta_B = 40.85^\circ
\]

\[R_B = 1.73P/\sin \theta_B = 2.64P \quad \text{or} \quad 2.0P/\cos \theta_B = 2.64P\]

Thus, \( R_B = 2.64P \) at 40.85° to the horizontal. All reactions are now known. Note that the reaction at \( B \) must be inclined so that it provides a resistive force to balance the horizontal component of the applied load. If the reactive force simply acted vertically, as in the previous example, the structure would not be in equilibrium. Note also that if there were two rollers present, instead of a pin and a roller, it would be impossible for the reactions to provide equilibrium restraints, and the structure would slide to the right.

In the moment calculations, several forces went through the moment center. Moment arms were thus of zero length, indicating that the forces produced no rotational effects about the point considered. In the future, forces having zero-length moment arms will not be included in the equilibrium calculations.

EXAMPLE

Determine the unknown reaction forces \( R_A \) and \( R_B \) in the structure shown in Figure 2.19.

**Figure 2.19**
Solution: The pinned connection on the right can provide a force $R_B$ that acts in any direction. Since there are no external forces acting horizontally, however, it is evident that the force must act only vertically, and $R_B = 0$. (If $R_B$ had a component in the $x$ direction, $\Sigma F_x = 0$ could not be satisfied.) The roller on the left can transmit a force $R_A$ in the vertical direction only, but this force may be upwardly or downwardly directed. Assume that it acts upwardly. Then we have the following calculations:

Moment equilibrium about $A$, $\Sigma M_A = 0$ \( \Phi \):

\[(R_B \times 10) - (4P \times 15) = 0 \quad \therefore \quad R_B = 6P\]

Find $R_A$ from $\Sigma F_y = 0$ \( \uparrow \downarrow \):

\[
\begin{align*}
R_A + R_B - 4P &= 0 \\
R_A + 6P - 4P &= 0 \\
R_A &= -2P \uparrow \quad \text{or} \quad +2P \downarrow 
\end{align*}
\]

The negative sign in the solution indicates that the direction assumed for $R_A$ was incorrect and that it actually acts in the direction shown (as a "tie-down" force). Rollers can be designed to act as tie-downs.

---

**EXAMPLE**

Determine the unknown reaction forces $R_A$ and $R_B$ in the structure shown in Figure 2.20.

![Figure 2.20](image)

Solution: This problem is similar to the last, but shows that the direction of the force at the roller is dependent on the magnitudes and locations of the loadings present.

Moment equilibrium about point $A$, $\Sigma M_A = 0$ \( \Phi \):

\[-6P(5) - 4P(15) + 10R_B = 0 \quad \therefore \quad R_B = 9P \uparrow\]

Force equilibrium of all the forces acting in the vertical direction, $\Sigma F_y = 0$: \( \uparrow \downarrow \):

\[
R_A + R_B - 6P - 4P = 0 \quad \therefore \quad R_A = 1.6P \uparrow
\]
EXAMPLE

Determine the unknown reaction forces $R_A$ and $R_B$ in the structure shown in Figure 2.21.

Figure 2.21

Solution: The pinned connection on the right can provide a force resistance in any direction, as is reflected in Figure 2.21 by an unknown force $R_B$ acting at the connection at an arbitrary angle. The roller on the left can transmit forces in the vertical direction only. Components of $R_B$ are initially used.

Force equilibrium of all the forces acting in the vertical direction, $\Sigma F_y = 0$ \(\uparrow\) +:

\[ R_A + R_{B_y} - 4P = 0 \quad \text{or} \quad R_A + R_{B_y} = 4P \]

Force equilibrium in the horizontal direction, $\Sigma F_x = 0$ \(\rightarrow\) :

\[ 2P - R_{B_x} = 0 \quad \therefore \quad R_{B_x} = 2P \]

Moment equilibrium about point A, $\Sigma M_A = 0$ \(\Phi\) :

\[ R_A(0) - (4P)(10) + R_{B_x}(20) + R_{B_y}(0) - (2P)(5) = 0 \quad \therefore \quad R_{B_y} = 2.5P \uparrow \]

Find $R_A$ from $\Sigma F_y = 0$: $R_A + R_{B_y} = 4P$:

\[ R_A + 2.5P = 4P \quad \therefore \quad R_A = 1.5P \uparrow \]

Find the resultant force $R_B$ from $R_{B_x}$ and $R_{B_y}$:

\[ R_B = \sqrt{R_{B_x}^2 + R_{B_y}^2} = 3.2P \]

\[ \tan^{-1}\theta = 2.5/2.0 \quad \therefore \quad \theta = 51.3^\circ \]

EXAMPLE

Determine the unknown forces $R_A$ and $R_B$ in Figure 2.22. The uniformly distributed load is first converted into an equivalent concentrated load for equilibrium calculations.

Solution: Moment equilibrium about A: $\Sigma M_A = 0$ \(\Phi\):

\[ \Sigma M = 0: \quad R_A L - (wL)(L/2) = 0 \]

\[ R_A = wL/2 = (50 \text{ lb/ft})(20 \text{ ft})/2 = 500 \text{ lb} \]
Figure 2.22  Uniformly loaded beam.

Force equilibrium in the vertical direction: \( \Sigma F_y = 0 \) \( \downarrow \uparrow \):

\[
\Sigma F = 0: \quad R_A + R_B - wL = 0 \quad \therefore \quad R_B = 500 \text{ lb}
\]

Note that by symmetry, \( R_A = R_B \). Thus, it is possible to solve for the unknown forces by considering equilibrium in the vertical direction only: \( \Sigma F = 0; \quad R_A + R_B = \text{total downward load} = wL \). Thus, \( R_A = R_B = wL/2 = 500 \text{ lb} \).

example

Determine the reactions for the structure shown in Figure 2.23. Note that 1 kip = 1000 lb.

Figure 2.23

Solution:  For the purposes of determining the reactions, the uniformly distributed load that acts over part of the structure is modeled as a statically equivalent concentrated load.

Moment equilibrium about point A, \( \Sigma M_A = 0 \):

\[
[(2 \text{ kips/ft})(10 \text{ ft})](15 \text{ ft}) - 20R_B \quad \therefore \quad R_B = 15 \text{ kips}
\]

Equivalent moment arm (to loading center).

Force equilibrium in the vertical direction, \( \Sigma F_y = 0 \):

\[
+R_A + R_B - [(2 \text{ kips/ft})(10 \text{ ft})] = 0 \quad \therefore \quad R_A = 5 \text{ kips}
\]
EXAMPLE

Determine the reactions for the structure shown in Figure 2.24. Use components.

\[ \theta = 30^\circ \]

**Figure 2.24** Beam with roller resting on a sloped support.

**Solution**

**Equilibrium in the vertical direction:** \( \Sigma F_y = 0: R_{Ay} + R_{By} - wL = 0 \).

The total load acting downward due to the distributed loading is simply the load per unit length multiplied by the length over which the load acts. Note that the direction of the unknown reaction \( R_B \) is initially known because roller joints can transmit loads only perpendicular to the surfaces on which they roll. Hence, \( R_B \) must be \( R_B \sin 60^\circ \). The direction of \( R_A \) is not known a priori.

**Moment equilibrium about point \( A \), \( \Sigma M_A = 0 \):**

\[ +R_{By}(L) - \left( wL \right) \left( \frac{L}{2} \right) = 0 \quad \text{or} \quad R_{By} = \frac{wL}{2} \]

Hence, \( R_{Ay} = wL/2 \) from \( \Sigma F_y = 0 \). All other unknown components of the two reactive forces pass through the moment center and consequently have zero moment arms and drop out of the equation. The moment produced by the uniformly distributed load was found by imagining it to be concentrated at its center of mass and finding the moment produced by this concentrated load about point \( A \). (See Section 2.2.7.)

**Final reactions:**

Since \( R_B \) is now known, \( R_B \) can be calculated next. Thus, \( R_{Ba} = R_B \sin 60^\circ \) or \( R_B = R_{Ba}/\sin 60^\circ = (wL/2)/\sin 60^\circ = 0.58wL \). \( R_{Bx} \) is simply \( R_B \cos 60^\circ \), or \( 0.29wL \). From \( \Sigma F_x = 0 \), it can be seen that \( R_{Bx} = R_{Ax} \). Hence, \( R_{Ax} = 0.29wL \). Since \( R_{Ay} = wL/2 \), it follows that \( R_A = 0.58wL \).
Principle of Superposition. For typical rigid structures that carry multiple loadings, it is possible to determine the reactions for each individual load and then to add together all of the results obtained. This approach is often useful for conceptualizing the behavior of structures under different loading conditions.

**EXAMPLE**

Determine the reactions to the structure in Figure 2.25, using a superposition technique.

![Figure 2.25 Principle of superposition.](image)

**Solution:** The reactions for each individual loading are determined first (as shown to the right in Figure 2.26) and are then simply added.

Forces and Reactions Associated with Overturning. In many situations when applied forces act horizontally or have significant horizontal components, they tend to cause a structure to overturn. Determining reactions for these situations is straightforward and no different in principle from determining reactions in any previous example. Care must simply be taken to use the correct moment arms associated with the overturning and reactive forces. The next example illustrates how a rigid structure resists overturning via the use of tie-down supports. The example after that illustrates how a structure with no tie-down supports can resist overturning via its own dead weight.

**EXAMPLE**

Determine the reactions for the structure shown in Figure 2.26.

![Figure 2.26](image)
Solution  Moment equilibrium about point A, \( \Sigma M = 0 \):  
\[-P(L) - 2P(L) + R_{B_y}(L) = 0, \quad \text{or} \quad R_{B_y} = 3P \uparrow\]

Equilibrium in the vertical direction, \( \Sigma F_y = 0 \):  
\[-R_{A_y} + R_{B_y} - 2P = 0; \quad \text{hence,} \quad R_{A_y} = P \downarrow\]

Equilibrium in the horizontal direction, \( \Sigma F_x = 0 \):  
\[-R_{A_x} + P = 0, \quad \text{or} \quad R_{A_x} = P \leftarrow\]

Resultant force at A:  
\[ R_A = 1.4P @ 45^\circ \]

Figure 2.27 illustrates the rotational stability analysis of a block with a large dead weight that is subjected to an overturning force. In this case, there are no active mechanisms such as pins to prevent overturning. As will be seen, when the dead weight is high, the applied overturning moment is less than the moment available to resist overturning (which is associated with the dead weight of the structure), and the structure is stable. If the applied moment is greater than the resisting moment, the structure overtops. If the applied and resisting moments are exactly equal, the structure is in neutral equilibrium. The example is a simplification of the buttress analysis shown in Figure 2.5(b). Many high-rise buildings, however, that have no tie-down piles, but that rely on their own dead weights and proportions to resist overturning due to wind or earthquake forces, can be analyzed similarly.

Block weight = \( W \)
Applied force = \( P \)

Stability conditions with respect to rotations about point A:
Overturning condition: \( Ph > Wd \)
Neutral condition: \( Ph = Wd \)
Stable condition: \( Ph < Wd \)

Graphical analysis

If \( P = 50 \text{ lb} \)
\( W = 200 \text{ lb} \)
\( h = 3 \text{ ft} \)
\( d = 1 \text{ ft} \)
\( Ph < Wd \)
\( (50) (3) < (200)(1) \)
Stable condition

If \( P = 100 \text{ lb} \)
\( W = 200 \text{ lb} \)
\( h = 3 \text{ ft} \)
\( d = 1 \text{ ft} \)
\( Ph > Wd \)
\( (100)(3) > (200)(1) \)
\([300 \cdot 200 = 100 \text{ ft-lb}] \)
Overturning tendency
Note that \( R = 223.6 \text{ lb} @ a = 0.45 \text{ ft} \)
= 100 ft-lb
Block rotates about A.

Figure 2.27  Block stability analysis with respect to point A.
General equilibrium conditions based on a moment analysis are shown at the top of the figure. If the overturning moment of the horizontal load exceeds the resisting moment associated with the weight and size of the block, then the block will overturn. If not, then the block is either in neutral equilibrium or stable. The lower part of the figure uses both an algebraic procedure and a graphically oriented approach similar to that shown in Figure 2.5. As long as the inclined force (which is statically equivalent to the horizontal and vertical forces combined) passes through the point of tipping or to its right, the block will not overturn. (Note that the block could be stable, but cracking might still occur in a masonry structure; see Section 7.3.1).

The example demonstrates that increasing resistance to overturning can be accomplished by either increasing the dead weight of the structure, increasing the width of the base footing, decreasing the height at which the applied horizontal force acts, or applying some combination of these techniques.

**Special Types of Reactions: Fixed-End Moments.** Many typical structures, especially cantilevered beams, have their ends rigidly attached to walls or other supports. The nature of this connection is such that it completely restrains the end of the member from either rotating or translating. Thus, a member can simply project from a wall or column face. Reactive forces that are developed at the support include the usual vertical and horizontal forces that prevent the member from translating, but also *restraining moments* that prevent the end of the member from rotating. These restraining moments, which balance the external applied moments at the same point, are typically called *fixed-end moments* and are a special form of reaction. One kind of restraining moment can be felt simply by extending one's arm and feeling what happens at the shoulder joint. Typically, these restraining moments are provided for by physical mechanisms that depend on the type of structure used. In a wide-flange beam cantilevered from a column, for example, the top and bottom flanges of the member are welded to the column face. Force couples are developed in the welds that provide the restraining fixed-end moment. (See Figure 16.3 in Chapter 16.) If a structure has multiple fixed ends (e.g., if it is fixed on both ends) or many other reaction points, it becomes *statically indeterminate*, and values cannot be found by the techniques presented in Chapter 2. (See Chapters 8 and 9 for solution techniques.)

**EXAMPLE**

Determine the fixed-end moments for the two beams in Figure 2.28.

![Fixed-end moment diagram](image)

**Solution:** For the beam to the left with the concentrated load \( P \), the rotational moment at support \( A \) that is associated with the applied load is simply given by \( M_{\text{applied}} = PL \). The balancing fixed-end moment is
equal in value, but of opposite sense; thus, $M'_k = PL$. For the beam to the right with the uniformly distributed loading, the rotational moment at support $A$ that is associated with the applied load is simply given by $M_{\text{applied}} = (wL)(L/2) = wL^2/2$. The balancing fixed-end moment is equal in value, but of opposite sense; thus, $M'_k = wL^2/2$.

**Cables.** Many structures are supported on one end or the other by cables. An example of a cable-supported structure is shown in Figure 2.29. A typical analytical objective would be to identify the forces in the cables and, ultimately, to determine their required diameters. A related objective would be to identify the reactive forces exerted by both the cables and the floor and roof systems on the adjacent main structure so that the cable can be properly sized. In a real structure, this process invariably involves first estimating dead and live loads and determining how those loads are finally carried by the framing structure and related cables. The general process of estimating loads and creating the necessary **loading model** is described in detail in Chapter 3. Briefly, in this example, roof loads are picked up by the facing cross beam, which in turn carries its loads to the corner columns. Forces are carried through the columns to the end cable-supported beams. Floor loads are also picked up by the lower cross beam and carried directly to the beam end. A highly simplified load model is shown in Figure 2.29. Once this model is developed, the principles of statics—the focus of this chapter—can be used to determine forces in the cables and connection points. Interestingly, the use of cables in the way shown normally means that horizontal as well as vertical forces are developed on the adjacent structure. Several examples of finding forces in cables are given subsequently.

![Diagram of cable-supported structure](image)

**Figure 2.29** Determination of the loading model for a cable-supported structure. The analysis is highly simplified and based on a number of assumptions.
Reactive forces are developed within the cables that provide equilibrium for the supported structure. As noted in Figure 2.15, the forces developed in a cable are in tension (cables cannot provide compressive force resistance) and inherently directed along the length of the cable. Thus, the direction of a cable defines the direction of the reaction it provides. The reactive force developed is equivalent to the internal tension force in the cable.

**EXAMPLE**

Determine the reactions to the cable-supported structure shown in Figure 2.30.

![Diagram of cable-supported structure](Diagram)

Canilever beam supported by a cable. An internal force \( T \) is developed within the cable. This force \( T \) can be found directly by summing moments about point \( A \).

\[
R_A = R_A \cos \phi = 0.866 P
\]

\[
R_A = R_A \sin \phi = 0.5P
\]

\[
T_{BC} = T_{BC} \cos 30 = 0.866 P
\]

\[
T_{AC} = T_{AC} \cos 30 = 0.866 P
\]

\[
T_{AC} = T_{AC} \sin 30 = 0.5P
\]

\[
R_A = 0.5P
\]

\[
R_A = 0.866 P
\]

\[
R_A = P \cos \phi = 0.5P
\]

Final equilibrium diagram showing final forces.

**Solution:** The steps in the determination of forces are shown in Figure 2.30. An equilibrium diagram is drawn first in which the direction of the reaction provided by the cable is shown as coincident with the location and direction of the cable. The internal force in the cable, \( T_{BC} \), is treated as an external reaction for calculating the equilibrium of the beam \( AB \). Moments are summed about point \( A \) to obtain \( T_{BC} \). Note that the moment arm of \( T_{BC} \) is perpendicular to the line of action of \( T_{BC} \). Once \( T_{BC} \) is determined, forces are summed in the horizontal direction to obtain the horizontal component of the left reactive force \( R_A \). (The horizontal component of the cable force is equal to the horizontal component of the left reaction.) Summing forces in the vertical direction yields the vertical component of \( R_A \). \( R_A \) itself can then be found from its components. A final equilibrium diagram is drawn. The three forces acting on the structure meet at a point. (See Section 2.3.2, on three-force members.) Note that the use of a cable induces a compressive force in beam \( AB \) equivalent to the horizontal component of the cable force. Note also that the reactive force at \( C \) is equal and opposite to the force in the cable.
EXAMPLE

A stayed-cable mast is shown under two different loading conditions in Figure 2.31. Determine the forces present in the cable and at the support points.

**EXAMPLE:**

**Force at top connection**

\[ \Sigma M_A = 5T_{BC} - 10(6000) \]
\[ 5T_{BC} = 10(6000) \]
\[ T_{BC} = 12000 \text{ lb} \]

\[ R_C = 12000 \cos 60^\circ \]
\[ = 5000 \text{ lb} \]

\[ R_A = 0 \]
\[ R_C = 10392 \text{ lb} \]

**EXAMPLE:**

**Force at midpoint of column**

\[ \Sigma M_A = 5T_{BC} - 5(6000) \]
\[ 5T_{BC} = 5(6000) \]
\[ T_{BC} = 6000 \text{ lb} \]

\[ R_C = 6000 \cos 60^\circ \]
\[ = 3000 \text{ lb} \]

\[ R_A = 6000 \sin 60^\circ \]
\[ = 5196 \text{ lb} \]

\[ R_C = 6000 \text{ lb} \]
\[ \theta = 60^\circ \]

**Figure 2.31** Cable-stayed column with two loading conditions.

**Solution:** The steps of the solution are shown in the figure. In the case of the first loading condition, where the force is applied at the top of the mast, the load \( P \) tends to produce a clockwise overturning moment that is balanced by a resisting moment associated with force \( T_{BC} \) developed in cable \( BC \). The cable force is assumed to be in tension, as it must be, and acts in the counter-clockwise direction. The moment arm for the cable force is the perpendicular distance from the base pin \( A \) to the cable \( BC \). The cable force \( T_{BC} \) is found by summing moments about \( A \). Thus, \[ 5T_{BC} = 10(6000) \], or \[ T_{BC} = 12000 \text{ lb} \]. Note that the reaction at \( C \) is numerically equal to the force \( T_{BC} \) developed in the cable. Equilibrium in the vertical and
horizontal directions is considered next to find the reactive force at A. Initially assumed to be inclined, this force turns out to act only vertically. Since the reaction is vertical (in this specific case), the compressive force developed in the mast has an equivalent numerical value. In the bottom example, the location of the applied horizontal force has been lowered and the same kind of analysis repeated. In this case, the reaction at A was found to have a horizontal component. The reactive force at A cannot simply be assumed to be oriented in the same way as is the mast. In this case, the force in the mast is numerically equivalent to the vertical component of the inclined reactive force at A. The mast is also subject to bending by force P.

2.3.4 Complete Static Analyses

The next example illustrates a complete static analysis for a cable-stayed structure. Reactions for the overall structure are determined first. The structure is then decomposed into its fundamental components, each of which is shown with the complete set of external and internal or reactive forces acting upon it. Reactions for one member become applied forces on the adjacent member (e.g., the reactions to the beam become equal and opposite forces applied to the mast). The equilibrium of different components can be considered in turn until all of the unknown forces at connections have been found. Note the way that forces at connections are shown as equal and opposite, a necessary condition that follows from the fact that these forces are reactive in nature and actually internal to the overall structure.

EXAMPLE

For the structure shown in Figure 2.32(a), determine the reactions at the base of the structure, the internal force present in the stabilizing cable CA, and the internal force in cable CE that supports the projecting member.

Figure 2.32 (a) Structure; (b) final analysis results.
For the Whole Structure:

Find $T_{CA}$ and $R_A$ (Figure 2.33):

Figure 2.33

\[ \sum M_B = 0: \]
\[ T_{CA}(h \sin \theta) - Pa = 0 \]
\[ T_{CA}(h \sin \theta) = Pa \]
\[ T_{CA}(8.45) = 5(1000) \]
\[ T_{CA} = 592 \text{ lb } @ \text{75\degree to horizontal} \]

Note that $R_A = T_{CA}$

\[ \sum F_x = 0: \]
\[ -R_A \sin 25\degree + R_{Bx} = 0 \]
\[ R_{Bx} = 250 \text{ lb} \]

\[ \theta = \tan^{-1}(1536/250) \]
\[ = 80.7\degree \text{ to horiz.} \]

\[ \sum F_y = 0: \]
\[ -R_A \cos 25\degree + R_{By} - 1000 = 0 \]
\[ R_{By} = 1536 \text{ lb} \]

For the Beam:

Find cable force $T_{CE}$ and $R_D$ (Figure 2.34):

Figure 2.34

\[ 7.07 \text{ ft} \]
\[ T_{CE} \]
\[ R_{Dx}, R_{Dy} \]
\[ 1000 \text{ lb} \]
\[ \Sigma M_D = 0: \quad T_{EC}[7.07] - 5(1000) = 0 \]
\[ T_{EC} = 707 \text{ lb} \]

Find \( R_{Dx} \) and \( R_{Dy} \):

\[ \Sigma F_x = 0: \quad -707 \cdot 2 \cos 45^\circ + R_{Dx} = 0, \quad \text{or} \quad R_{Dx} = 500 \text{ lb} \]
\[ \Sigma F_y = 0: \quad -707 \cdot 2 \sin 45^\circ + R_{Dy} = 0 \quad \text{or} \quad R_{Dy} = 500 \text{ lb} \]

Thus, \( R_D = \frac{500}{\sin 45^\circ} = 707 \text{ lb at } 45^\circ \)

Final equilibrium diagrams are shown in Figure 2.32(b). All members should be in equilibrium. (Check to make sure that \( \Sigma F_x = 0 \) and \( \Sigma F_y = 0 \).)

2.4 Forces and Moments

Forces and moments can be either external or internal. Forces or moments that are applied to a structure (e.g., a weight attached to the end of a rope) are described as external. Forces and moments that are developed within a structure in response to the external force system that is present in the structure (e.g., the tension in a rope resulting from the pull of an attached weight) are described as internal. Internal forces and moments are developed within a structure due to the action of the external force system acting on the structure. Such forces serve to hold together, or maintain the equilibrium of, the constituent particles or elements of the structure.

These internal forces or moments that are developed within a member are in direct response to externally applied forces and are typically in tension, compression, shear, or bending. They are rarely constant throughout a specific structure, but vary from point to point. Only in a simple structure, such as a rope with a weight on the end, are the forces constant within the structure. In a typical beam, however, the external forces acting on the structure generate internal forces that vary from point to point. This section begins a general study of these internal forces and their distributions by first considering simple members in a state of pure tension or compression, in which the external forces are applied along the length of the structure (so-called axial forces). Following this treatment is a series of sections on shear forces and bending moments that are developed within a structure. An ability to determine their magnitudes and distributions is fundamental to analyzing and designing structures. These axial forces, shear forces, and bending moments in turn produce stresses and corresponding deformations within the material fabric of the structure. Section 2.5 introduces the concept of stress. For now, we will focus on how to determine internal force and moment states in structures.

2.4.1 Axial Forces (Tension and Compression)

Consider the system shown in Figure 2.35. It is obvious that a tension force has developed in the cable supporting the block. This force has a magnitude equal to the weight of the block. As the diagram illustrates, the equilibrium of the block is maintained by the development of an internal force \( F_t \), in the cable. In this case, \( F_t \) is simply equal to the weight of the block.

Internal compressive forces are obviously similar in character, but opposite in sense. Figure 2.35 illustrates a member in which the internal compressive force varies due to the